

Do stochastic inhomogeneities affect dark-energy precision measurements?

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The effect of a stochastic background of cosmological perturbations on the luminosity-redshift relation is computed to second order through a recently proposed covariant and gauge-invariant light-cone averaging procedure. The resulting expressions are free from both ultraviolet and infrared divergences, implying that such perturbations cannot mimic a sizable fraction of dark energy. Different averages are estimated and depend on the particular function of the luminosity distance being averaged. The energy flux, being minimally affected by perturbations at large z , is proposed as the best choice for precision estimates of dark-energy parameters. Nonetheless, its irreducible (stochastic) variance induces statistical errors on $\Omega_\Lambda(z)$ typically lying in the few-percent range.

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Establishing the existence of dark energy and determining its parameters is one of the central issues in modern cosmology. Evidence for a sizable dark-energy component in the cosmic fluid comes from different sources: CMB anisotropies, models of large-scale-structure formation and, most directly, the luminosity redshift relation of Type Ia supernovae, used as standard candles.

In this latter case, on which we concentrate our attention, the analysis is usually made in the simplified context of a homogeneous and isotropic (FLRW) cosmology. The issue has then been raised about whether inhomogeneities may affect the conclusion of such a naive analysis. Inhomogeneous models in which we occupy a privileged position in the Universe, for instance, can mimic dark energy (as first pointed out in [1]), but look both unrealistic and highly fine-tuned. More interestingly, we should address this question in the presence of stochastically isotropic and homogeneous perturbations of the kind predicted by inflation. We present here the main ideas and results of such a study, leaving to a forthcoming paper [2] their detailed derivation and discussion.

There is by now general agreement that super-horizon perturbations cannot mimic dark-energy effects [3]. By contrast, the impact of sub-horizon perturbations is still unsettled [4–6] owing to the appearance of ultraviolet divergences while computing their “backreaction” on certain classes of large-scale averages [5, 6]. The possibility that these effects may simulate a substantial fraction of dark energy, or that they may at least play some role in the context of near-future precision cosmology, has to be seriously considered.

In order to address these issues we have studied the luminosity-redshift relation in a spatially-flat Λ CDM

model, perturbed by a stochastic background of inhomogeneities. The luminosity distance d_L now depends on the redshift z as well as on the angular coordinates of the sources, and must be inserted in an appropriate light-cone and *ensemble* average [7, 8]. Unlike for the cases treated in [5, 6], we find a result free of ultraviolet divergence and with no significant infrared contributions either. As a consequence, corrections are typically small, certainly too small to mimic a sizable fraction of dark energy. However, interestingly enough, both their size and their z -dependence strongly depend on the particular function of d_L being averaged.

We find, in particular, that the energy flux $\Phi \sim d_L^{-2}$ is practically unaffected by inhomogeneities, while the most commonly used variables (like the distance modulus $\mu \sim 5 \log_{10} d_L$) may receive much larger corrections. This creates (at least in principle) intrinsic ambiguities in the measure of the dark-energy parameters, unless the backreaction of stochastic inhomogeneities is properly taken into account.

Let us start by recalling the standard expression for the luminosity distance in an unperturbed flat Λ CDM model, with present fractions of critical density Ω_Λ and $\Omega_m = 1 - \Omega_\Lambda$:

$$d_L^{FLRW}(z) = \frac{1+z}{H_0} \int_0^z \frac{dz'}{[\Omega_\Lambda + \Omega_m(1+z')^3]^{1/2}}. \quad (1)$$

Consider now the expression for d_L in the corresponding perturbed geometry. Combining light-cone and ensemble averages (denoted, respectively, by brackets and over-bars), we can write the averaged result in the form:

$$\overline{\langle d_L \rangle}(z) = d_L^{FLRW} [1 + f_d(z)], \quad (2)$$

where $f_d(z)$ represents the “backreaction” on d_L due to inhomogeneities. For consistency, d_L has to be computed (at least) up to the second perturbative order since *ensemble* averages of first-order quantities are vanishing for stochastic perturbations. A detailed computation of $f_d(z)$ would thus enable to extract the “true” value of the dark-energy parameters from the measurement of $\langle d_L \rangle(z)$ after taking the correction into account.

However, as already stressed in [8], given the covariant (light-cone) average of a perturbed (inhomogeneous) observable S the average of a generic function of this observable differs, in general, from the function of its average, i.e. $\langle F(S) \rangle \neq F(\langle S \rangle)$. Expanding the observable to second order as $S = S_0 + S_1 + S_2 + \dots$, one finds:

$$\overline{\langle F(S) \rangle} = F(S_0) + F'(S_0) \overline{\langle S - S_0 \rangle} + F''(S_0) \overline{\langle S_1^2/2 \rangle}. \quad (3)$$

Thus different functions of the luminosity distance are differently affected by the inhomogeneities, and require different “subtraction” procedures. Finding the function that minimizes the backreaction will help of course for a precision estimate of the cosmological parameters. One of the main claims of this paper is the identification of such an optimal observable with the energy flux $\Phi = L/(4\pi d_L^2)$ received from a standard candle of luminosity L located on the observer’s past light-cone. We now illustrate how we have performed such a calculation.

The average value of Φ , obviously controlled by the average of d_L^{-2} , has to be carried out on the past light-cone of the observer, at a fixed redshift z , using the gauge-invariant prescription introduced in [7]. This is most conveniently done [7, 8] in the so-called geodesic light-cone gauge, where the metric depends on six arbitrary functions ($\Upsilon, U^a, \gamma_{ab}$, $a, b = 1, 2$), and the line-element takes the form (with $\tilde{\theta}^1 = \tilde{\theta}$, $\tilde{\theta}^2 = \tilde{\phi}$):

$$ds^2 = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab} (d\tilde{\theta}^a - U^a dw) (d\tilde{\theta}^b - U^b dw). \quad (4)$$

In this gauge the past light-cone is defined by the condition $w = w_0 = \text{const}$, and the redshift is given by:

$$1 + z = \Upsilon(w_0, \tau_0, \tilde{\theta}^a) / \Upsilon(w_0, \tau, \tilde{\theta}^a). \quad (5)$$

Furthermore, the luminosity distance of the source is simply expressed as [8] $d_L = (1+z)^2 \gamma^{1/4} (\sin \tilde{\theta})^{-1/2}$, yielding the following exact result:

$$\langle d_L^{-2} \rangle(z, w_0) = \frac{4\pi(1+z)^{-4}}{\int d^2 \tilde{\theta}^a \sqrt{\gamma(w_0, \tau(z, \tilde{\theta}^a), \tilde{\theta}^b)}}, \quad (6)$$

where $\gamma = \det \gamma_{ab}$, and $\tau(z, \tilde{\theta}^a)$ is obtained by solving Eq. (5). The above expression has a simple physical interpretation: the averaged flux, for a given z , is inversely proportional to the proper area (computed with respect to the metric (4)) of the surface lying on our past light-cone at the given value of z . Flux conservation is probably at

the basis of the particular simplicity of this average and to its minimal deviation from the homogeneous value.

To compute this quantity in the perturbed geometry of our interest, we need to express it in a gauge where the stochastic background of cosmological perturbations is explicitly known up to second order. To this purpose, we can use the standard Poisson gauge where we include first and second-order scalar perturbations, neglecting their tensor and vector counterparts (see [2] for a discussion of this point). Performing the relevant transformations to second order we arrive at the following analogue of (2):

$$\overline{\langle d_L^{-2} \rangle} = (d_L^{FLRW})^{-2} (\overline{I_\Phi(z)})^{-1} \equiv (d_L^{FLRW})^{-2} [1 + f_\Phi(z)], \quad (7)$$

where I_Φ has in general the following structure:

$$I_\Phi(z) = \int \frac{d\phi d\theta \sin \theta}{4\pi} [1 + \mathcal{I}_1 + \mathcal{I}_{1,1} + \mathcal{I}_2](\theta, \phi, z). \quad (8)$$

Here \mathcal{I}_1 , $\mathcal{I}_{1,1}$, \mathcal{I}_2 are, respectively, the first-order, quadratic first-order, and genuine second-order contributions of our stochastic fluctuations. After solving the relevant perturbation equations [9] they can all be expressed in terms of the first-order Bardeen potential $\Psi(x, \eta)$. Using the stochastic properties of this perturbation, and expanding in Fourier modes $\Psi_k(\eta)$, we can then obtain an expression for $\overline{(I_\Phi)^{-1}}$ where first-order contributions drop out because of the *ensemble* average, and the scalar perturbations only appear through the so-called dimensionless power spectrum, $\mathcal{P}(k, \eta) = (k^3/2\pi^2) |\Psi_k(\eta)|^2$.

Unfortunately, $\overline{(I_\Phi)^{-1}}$ contains integrals over null geodesics lying on the past light-cone of the given observer [8], which get intertwined with the time-dependence of \mathcal{P} , forcing us to proceed with an approximate numerical integration. This will be done below, after inserting (as an instructive example) an illustration of the limiting CDM case, where all integrals but the one over k can be done analytically thanks to the time-independence of \mathcal{P} [8].

In that case the result can be written in the form

$$f_\Phi(z) = \int_0^\infty \frac{dk}{k} \mathcal{P}(k) [f_{1,1}(k, z) + f_2(k, z)], \quad (9)$$

where $f_{1,1}$ and f_2 are complicated –but known– analytic functions of their arguments [2]. Furthermore, the leading contribution in the region of z relevant for dark-energy phenomenology comes from terms of the type $f(k, z) \sim (k/\mathcal{H}_0)^2 \tilde{f}(z)$, where \mathcal{H}_0 is the present Hubble scale. We can then write, to a very good accuracy,

$$f_\Phi(z) \simeq [\tilde{f}_{1,1}(z) + \tilde{f}_2(z)] \int_0^\infty \frac{dk}{k} \left(\frac{k}{\mathcal{H}_0} \right)^2 \mathcal{P}(k), \quad (10)$$

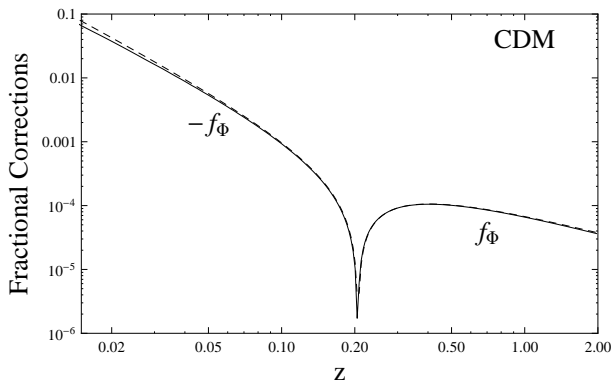


FIG. 1: The fractional correction f_Φ of Eq. (9) (solid curve), compared with the same quantity given to leading order by Eq. (10) (dashed curve), in the context of an inhomogeneous CDM model. We have used for $\mathcal{P}(k)$ the inflationary scalar spectrum with the WMAP parameters [10] and the transfer function given in [11] (see also [8]). The plotted curve refers, as an illustrative example, to an UV cutoff $k_{UV} = 1\text{Mpc}^{-1}$.

where an explicit calculation gives [2]:

$$\tilde{f}_{1,1}(z) = \frac{10 - 12\sqrt{1+z} + 5z(2 + \sqrt{1+z})}{27(1+z)(\sqrt{1+z}-1)^2}, \quad (11)$$

$$\tilde{f}_2(z) = -\frac{1}{189} \left[\frac{2 - 2\sqrt{1+z} + z(9 - 2\sqrt{1+z})}{(1+z)(\sqrt{1+z}-1)} \right]. \quad (12)$$

The absolute value (and sign) of $f_\Phi(z)$ are illustrated in Fig. 1, showing the accuracy of the leading order terms (10) and confirming that the backreaction of a realistic spectrum of stochastic perturbations induces negligible corrections to the averaged flux at large z (the larger corrections at small z , due to “Doppler terms”, has been discussed in [8]). In addition, it shows that, in any case, such corrections have the wrong z -dependence (in particular change sign at some z) to simulate even a tiny dark-energy component. For the considered spectrum (behaving as $k^{n_s-5} \log^2 k$ at large k , see [11]) the spectral integral is convergent and very weakly sensitive to the chosen value of the UV cutoff [8] representing here the limit of validity of our perturbative approach.

We now come to the more realistic Λ CDM case, where the f_Φ correction should be obtained by a full numerical integration of Eqs. (7,8). For simplicity, we will only take into account those terms giving the leading (k^2 -enhanced) contributions in the CDM case. For Λ CDM we can generally expect a smaller correction due to the fact that the spectrum is now suppressed, at large k , by a lower value of the equality scale k_{eq} [11]. This is confirmed by the explicit numerical result for $|f_\Phi|$ presented in Fig. 2. The small value of $|f_\Phi|$ at large z leads us to conclude that the averaged flux is a particularly appropriate quantity for extracting from the observational data the “true” cosmological parameters. As we are go-

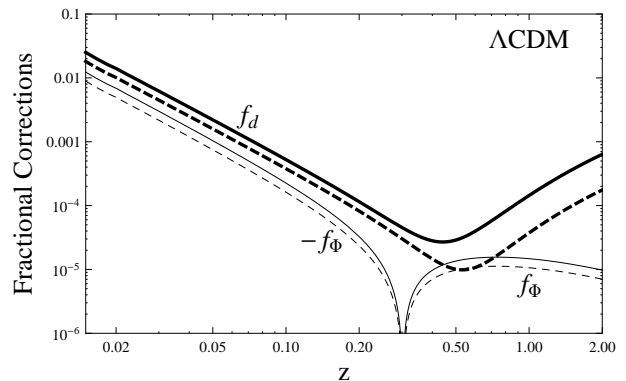


FIG. 2: The fractional correction to the flux f_Φ of Eq. (7) (thin curves) is compared with the fractional correction to the luminosity distance f_d of Eq. (13) (thick curves), for a Λ CDM model with $\Omega_\Lambda = 0.73$. We have used two different cutoff values: $k_{UV} = 0.1\text{Mpc}^{-1}$ (dashed curves) and $k_{UV} = 1\text{Mpc}^{-1}$ (solid curves); the spectrum is the same as that of Fig. 1, adapted to Λ CDM.

ing to see now, the situation is somewhat different for other functions of d_L .

Indeed, let's apply the general result (3) to the flux variable, $S = \Phi$, and consider two important examples: $F(\Phi) = \Phi^{-1/2} \sim d_L$, and $F(\Phi) = -2.5 \log_{10} \Phi + \text{const} \sim \mu$ (the distance modulus). For the luminosity distance, following the notations of Eq. (2) and using the general result (3), we obtain:

$$f_d = -(1/2)f_\Phi + (3/8)\overline{(\Phi_1/\Phi_0)^2}. \quad (13)$$

Similarly, for the distance modulus we obtain:

$$\overline{\langle \mu \rangle} - \mu^{FLRW} = -1.25(\log_{10} e) \left[2f_\Phi - \overline{(\Phi_1/\Phi_0)^2} \right], \quad (14)$$

where f_Φ is defined in Eq. (7).

As clearly shown by the two above equations, the corrections to the averaged values of d_L and μ are qualitatively different from those of the flux (represented by f_Φ), because of the extra contribution (inevitable for any non-linear function of the flux) proportional to the square of the first-order fluctuations. As mentioned before, the averaged flux corrections have leading spectral contributions of the type $k^2\mathcal{P}(k)$; on the contrary, the new corrections to d_L and μ are due to the so-called “lensing effect”, dominate at large z , and have leading spectral contributions of the type $k^3\mathcal{P}(k)$ (as already discussed in [8]). The explicit numerical integration, reported in Fig. 2, confirms that, as a result, $|f_\Phi| \ll f_d$ at large z . We stress that even the k^3 -enhanced contributions are UV-finite for the case under consideration.

Let us now briefly discuss to what extent the enhanced corrections due to the squared first-order fluctuations can affect the determination of the dark-energy parameters if quantities other than the flux are used in the

fits. To this purpose we consider the much used (average of the) distance modulus given in Eq. (14), referred as usual to the homogeneous Milne model with $\mu^M = 5 \log_{10}[(2+z)z/(2H_0)]$. In Fig. 3 we compare the averaged value $\langle \mu \rangle - \mu^M$ with the corresponding expression in a homogeneous Λ CDM model with different values of Ω_Λ . We also show the expected dispersion around the averaged result, represented by the square root of the variance [8]. The latter is given by:

$$\sqrt{\langle \mu^2 \rangle - \langle \mu \rangle^2} = \pm 2.5(\log_{10} e) \sqrt{\langle (\Phi_1/\Phi_0)^2 \rangle}; \quad (15)$$

while for the flux we simply find:

$$\sqrt{\langle (\Phi/\Phi_0)^2 \rangle - \langle \Phi/\Phi_0 \rangle^2} = \pm \sqrt{\langle (\Phi_1/\Phi_0)^2 \rangle}. \quad (16)$$

As illustrated in Fig. 3, we find that, even for the distance modulus, the effect of inhomogeneities on the average only affects the determination of Ω_Λ at the third decimal figure (see also Fig. 2 and [2]), at least for the inflationary power spectrum with the Λ CDM transfer function of [11]: in that case, the curves for $\langle \mu \rangle$ and μ^{FLRW} are practically coincident at large z . We have considered other spectra which take into account non-linear effects and have more power at short scales, like those obtained following [12]. Using such spectra only affects very mildly the k^2 -enhanced terms (hence the flux) while they increase the corrections wherever the k^3 -enhanced lensing terms play a major role. In particular, the variance due to the fluctuations, which is already at the few-% level at large z for the power spectrum of [11] (see Fig. 3), can be further increased [2]. Note that, even for these improved spectra, all our integrals are still free of UV divergences since, in any case, \mathcal{P} falls faster than k^{-3} (i.e. the matter density contrast spectrum grows slower than k).

Our main conclusions can be summarized as follows:

(1) Unlike in the case of other large-scale averages [5, 6], a physically motivated light-cone averaging of the luminosity-redshift relation leads to expressions for the backreaction of stochastic inhomogeneities that are free from UV (as well as IR) divergences. This allows us to conclude that such perturbations cannot simulate a substantial fraction of dark energy.

(2) The actual value of the backreaction strongly depends on the quantity being averaged. It turns out to be minimal for the flux Φ , which is also practically insensitive to the short-distance behaviour of the power spectrum. Therefore, the flux stands out as the safest observable for precision cosmology. For other observables, such as the distance modulus, the backreaction is considerably larger and is more sensitive to the spectrum used.

(3) The dispersion due to stochastic fluctuations is much larger than the backreaction itself, implying an irreducible scatter of the data that may limit to the percent level (see Fig. 3) the precision attainable on cosmological parameters because of the present limited statistics.

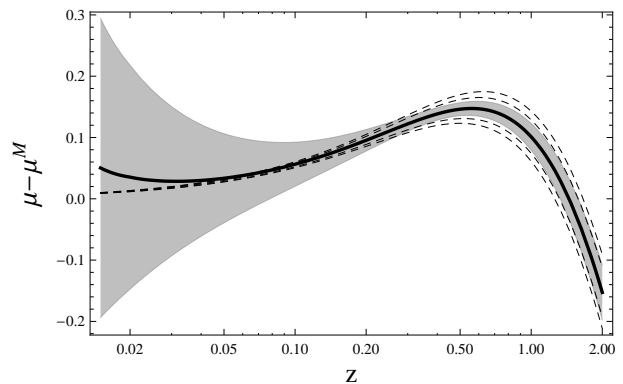


FIG. 3: The averaged distance modulus $\langle \mu \rangle - \mu^M$ (thick solid curve), and its dispersion of Eq. (15) (shaded region) are computed for $\Omega_\Lambda = 0.73$ and compared with the homogeneous value for the unperturbed Λ CDM models with $\Omega_\Lambda = 0.69, 0.71, 0.73, 0.75, 0.77$ (dashed curves). We have used $k_{UV} = 1 \text{ Mpc}^{-1}$ and the same spectrum as in Fig. 2.

(4) We calculated here the full second order effect of stochastic perturbations. Possible contributions coming from the non-perturbative regime on length scales much smaller than 1 Mpc have still to be taken in consideration before final conclusions can be drawn.

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